Effective action for bubble nucleation rates

Ian G. Moss and Wade Naylor
Department of Physics, University of Newcastle Upon Tyne, NE1 7RU U.K.

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Abstract

We develop a method to calculate the prefactor in the expression for the bubble nucleation rate. A fermion with Yukawa coupling is considered where a step potential can be used as a good approximation in the thin wall limit. Corrections due to thicker walls are considered by perturbing about the thin wall case. We derive the thermal one loop effective action, calculating it numerically and find that the prefactor in the nucleation rate can both suppress and enhance.

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I. INTRODUCTION

Bubble nucleation can occur in a first order phase transition from the false to the true vacuum. The bubble nucleation rate per unit volume per unit time, written in the language of Coleman [1], is $\Gamma/V = A\, e^{-B}$, where B is the classical euclidean action of the bubble and A is the one loop contribution including zero and negative modes.

The purpose of this paper is to present a method enabling one to calculate the nucleation rate in the thin wall limit. Techniques such as the derivative expansion break down for this type of background [6]. The thin wall limit is a possible scenario in electroweak theory with multiple Higgs fields [2]. It is also useful as a possible explanation for the generation of the baryon asymmetry we observe today [3]. In the interests of brevity we focus on just the fermion fields. However, the general discussion includes scalar and spinor fields for completeness. The fermion case has boundary conditions (see appendix) which are similar to those used in the MIT Bag model [4]. The calculation for the other fields is currently under way [5].

We briefly mention the work that has been done before on calculating the prefactor A. One method invlolves using the derivative expansion as an approximation, for example in electroweak theory [6]. Another method uses a result of Coleman's [7] to calculate the fluctuation determinant. This was done at finite temperature for ϕ^4 theory [8], electroweak theory [9] and for fermions [10], all for walls of finite thickness. Here we also develop an exact numerical scheme for working out the fluctuation corrections. Garriga [11] has calculated the prefactor for ϕ^4 theory, with an infinitely thin wall at zero and finite temperature using analytic methods. Corrections from thicker walls were also considered.

Lee [12] use a phase shift method similar to the one we shall employ (using some old results due to Schwinger [13]). They use a momentum cut off to regularise ϕ^4 theory, for thin and thick walls. In our method we shall use phase shifts and relate the prefactor to the heat kernel. The theory is regulated by subtracting off the relevant heat kernel coefficients. We use fermions with a Yukawa coupling and a step function profile to begin with. Results are at finite temperature, though the method can be easily extended to the zero temperature case.

The paper is organised as follows. In § 2A we relate the phase shift to the heat kernel and zeta function. In § 2B we calculate the thermal effective action. In § 3 we discuss how to consider corrections from thicker walls. In § 4 results are presented. In § 5 we draw conclusions and § 6 contains an appendix, with details on the calculation for the fermion phase shift.

II. HEAT KERNELS AND PHASE SHIFTS

A. Nucleation & Regularisation

We begin with the nucleation rates for the decay of a false vacuum at finite temperature due to an instanton ϕ_{bubble}. Any field that aquires a mass on the instanton background can contribute to the prefactor. In three dimensions [14] (see also comments in [12] & [15]),

$$A = T\left(\frac{B}{2\pi}\right)^{3/2}\prod_{fields}\left|\frac{det'[-\nabla^2+m^2(\phi_{bubble})]}{det[-\nabla^2+m^2(\phi_{sym})]}\right|^{-1/2}. \tag{1}$$

Three zero eigenvalues, arising from breaking the Poincare symmetry, each contribute $(B/2\pi)^{1/2}$ to the total and these are omitted from the scalar determinant, as indicated by the prime.

In the thin wall limit we assume that ϕ_{bubble} has a bubble wall at some radius R, such that ϕ takes the false vacuum value at radii $r > R$ and the true vacuum value at radii $r < R$, with a narrow transition region near $r = R$. For scalar bosons and fermions (and if calculated, the vector bosons), the relevant mass terms vanish in the false vacuum and are non-zero in the true vacuum, leading to a step function profile,

$$m(r) = \begin{cases} m & r < R \\ 0 & r > R \end{cases}. \tag{2}$$

In the same limit, the scalar Higgs field masses differ little for large and small radii. This suggests that the Higgs contribution to the prefactor is smaller than the fermion contribution (and also any other fields). We will consider the accuracy of the thin-wall approximation later.

The eigenvalues in the determinant can be found by using a partial wave analysis and phase shifts [16]. We first discretize the eigenmodes by putting them in a sphere of large radius Ω. After seperating the eigenmodes into radial functions and spherical harmonics, the radial parts asymptotically approach trigonometric functions of $kr + \phi$, where ϕ is a constant phase depending on k and the angular momentum l. On the boundary,

$$k_n\Omega \approx n\pi - \phi. \tag{3}$$

In the false vacuum, the potential is zero and we label the free eigenvalues $k_n^{(0)}$

$$k_n^0\Omega \approx n\pi - \phi^{(0)}. \tag{4}$$

On letting $\Omega \to \infty$ (continuum limit), the above equations (3) and (4) imply a relationship between the phase shift $\delta_l(k)$, the densities of states $g_l(k)$ and $g_l^0(k)$, [13],

$$g_l(k) = g_l^0(k) + \frac{1}{\pi}\frac{d\delta_l(k)}{dk}, \tag{5}$$

for the radial modes.

The difference between the heat kernels for the instanton and the true vacuum will be

$$\Delta K(t) = \sum_n\left(e^{-k_n^2 t} - e^{-k_n^{(0)2}t}\right) \tag{6}$$

Using the density of states factor $g_l(k)$ and the degeneracy factor χ_l,

$$\Delta K(t) = \int_0^\infty dk\, e^{-k^2 t}\sum_l \chi_l(g_l(k) - g_l^0(k)). \tag{7}$$

Substituting (5) into the above equation and integrating by parts we obtain

$$\Delta K(t) = \frac{2}{\pi}\int_0^\infty dk\, e^{-k^2 t}\, kt \sum_l \chi_l \delta_l(k), \tag{8}$$

where the degeneracy factor $\chi_l = (2l+1)$ in three dimensions.

The heat kernel can now be used to regularise the determinants appearing in the prefactor A in the nucleation rate. We define the generalised ζ function by

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\, \mathbf{tr}\, K(t)\, dt, \tag{9}$$

The analytic continuation of $\zeta(s)$ then gives

$$\log A = \sum_{fields} (\pm)\Delta W \tag{10}$$

where we take $\pm$ for scalar and spinor fields respectively, and

$$\Delta W = \tfrac{1}{2}\zeta'(0) + \tfrac{1}{2}\zeta(0)\log\mu^2 \tag{11}$$

where μ is the renormalisation scale.

For numerical work, the analytic continuation can best be performed by subtracting terms from the heat kernel. As $t \to 0$, the heat kernel in $d+1$ dimensions has the asymptotic expansion [19]

$$K(t) \sim t^{-(d+1)/2} \sum_{n=0} B_n t^n\,. \tag{12}$$

The leading terms, which cause the poles in the ζ function, can be removed by replacing the sum over phase shifts in equation (8) by

$$\sum_l \chi_l \bar{\delta}_l = \sum_l \chi_l \delta_l - \frac{\pi B_1\, k^{d-1}}{\Gamma(\frac{d+1}{2})} - \frac{\pi B_{3/2}\, k^{d-2}}{\Gamma(\frac{d}{2})} - \frac{\pi B_2\, k^{d-3}}{\Gamma(\frac{d-1}{2})}, \tag{13}$$

An infra-red cuttof M_{IR} must also be included, noting as we shall see later that the dependence on M_{IR} is illusory, since changing M_{IR} does not affect the value of ΔW.

For the step potential, standard expressions for the heat kernel coefficients give [19]:

$$B_1 = -\frac{m^2 R^{d+1}}{2^d (d+1)\Gamma(\frac{d+1}{2})}\,, \tag{14}$$

$$B_{3/2} = \frac{m^2 R^d}{2^d \Gamma(\frac{d+1}{2})}\,, \tag{15}$$

$$B_2 = \frac{m^4 R^{d+1}}{2^{d+1}(d+1)\Gamma(\frac{d+1}{2})}\,. \tag{16}$$

For example, the phase shift [16] for a scalar boson field is

$$\tan\,\delta_l = \frac{k\, J_{l-1/2}(kR)\, J_{l+1/2}(k'R) - k'\, J_{l-1/2}(k'R)\, J_{l+1/2}(kR)}{k\, N_{l-1/2}(kR)\, J_{l+1/2}(k'R) - k'\, J_{l-1/2}(k'R)\, N_{l+1/2}(kR)}\,, \tag{17}$$

where $k' = \sqrt{k^2 - m^2}$ and $l = 0, 1, ...$ in three dimensions. The phase shift for fermions is a rather lengthy calculation which is left until the appendix.

B. Thermal Effective Action

Using the techniques of the last section we are now ready to calculate what is essentially the difference in the effective action for the true and false vacuum ΔW. We refer the reader to [20] for a detailed disscusion of heat kernel methods at finite temperaure for scalar and spinor fields. The thermal heat kernel K^β can be expressed as an infinite sum of zero temperature heat kernels

$$K^\beta(t \mid \tau, x;\, \tau', x') = \sum_{n=-\infty}^{\infty} (\pm)^n\, K(t \mid \tau, x;\, \tau' + \beta n, x') \tag{18}$$

(where $\pm$ is for scalar and spinor fields respectively) and for ultrastatic spacetimes the heat kernel can be factorised into temporal and spacial parts giving

$$K(t \mid \tau, x; \tau', x') = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(\tau-\tau')^2}{4t}} K^{(3)}(t \mid x, x'). \tag{19}$$

It is then possible to show using the above relations that,

$$K^\beta(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2\beta^2}{4t}} K^{(3)}(t). \tag{20}$$

ΔW is related to the thermal heat kernel $\Delta K^\beta(t)$ for scalar fields by

$$\Delta W = -\frac{1}{2}\int_0^\infty \frac{dt}{t}\, \mathbf{tr}\Delta K^\beta(t). \tag{21}$$

Substituting (20) into the above equation, we obtain for scalars,

$$\Delta W = -\frac{\beta}{2}\int_0^\infty \frac{dt}{t} \sum_{n=-\infty}^{\infty} e^{-\frac{\beta^2}{4t}n^2} \frac{1}{\sqrt{4\pi t}} \frac{2}{\pi}\int_0^\infty dk\, e^{-k^2 t}\, k\, t \sum_{l=0}^{\infty} (2l+1)\, \delta_l(k). \tag{22}$$

Then, using the fact that

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\frac{\beta^2}{4t}n^2 - k^2 t} dt = \frac{\sqrt{\pi}}{k} e^{-n\beta k}, \tag{23}$$

we get

$$\Delta W = -\frac{\beta}{2\pi}\int_0^\infty dk \sum_{l=0}^{\infty}(2l+1)\, \delta_l(k) - \frac{\beta}{\pi}\int_0^\infty dk \sum_{l=0}^{\infty}(2l+1)\, \delta_l(k) \sum_{n=1}^{\infty} e^{-n\beta k}. \tag{24}$$

The sum over n is standard, leading to the result

$$\Delta W = -\frac{\beta}{2\pi}\int_0^\infty dk \sum_{l=0}^{\infty}(2l+1)\, \bar{\delta}_l(k) - \frac{\beta}{\pi}\int_0^\infty dk \sum_{l=0}^{\infty}(2l+1)\, \frac{\delta_l(k)}{e^{\beta k} - 1}. \tag{25}$$

The first term is the zero point energy, which contains the ultraviolet divergences of the theory and hence $\bar{\delta}_l(k)$ (see Eq.20 & 13). The second term is the temperature dependant part. The above expression can be derived using densities of states methods [12]. Thus, upon regulating the non-thermal part, using zeta function regularization and introducing a mass which we let tend to zero at the end of the calculation, we have

$$\Delta W^N = -\frac{\beta}{2\pi}\int_0^\infty dk\left(\sum_{l=0}^{\infty}(2l+1)\delta_l(k) - 2\sqrt{\pi}B_1 k - \frac{\sqrt{\pi}B_2}{\sqrt{(k^2+M_{IR}^2)}}\right) + \frac{\beta B_2}{2\sqrt{4\pi}}\, logM_{IR}^2, \quad (26)$$

$$\Delta W^T = -\frac{\beta}{\pi}\int_0^\infty dk \sum_{l=0}^{\infty}(2l+1)\,\frac{\delta_l(k)}{e^{\beta k}-1}, \quad (27)$$

where $\Delta W = \Delta W^N + \Delta W^T$ is the thermal effective action and we are working explicitly in three dimensions. (Note that ΔW is independent of M_{IR}^2 and for the scalar boson $B_{3/2}$ is zero.)

For fermions the spinor effective action $\Delta W_{(1/2)}$ is related to the heat kernel $\Delta K^{\beta}_{(1/2)}(t)$ by

$$\Delta W_{(1/2)} = \int_0^\infty \frac{dt}{t}\, \mathrm{tr} K^{\beta}_{(1/2)}(t). \quad (28)$$

Thus,

$$\Delta W_{(1/2)} = 4\beta \int_0^\infty \frac{dt}{t} \sum_{n=-\infty}^{\infty}(-1)^n e^{-\frac{\beta^2}{4t}n^2}\frac{1}{\sqrt{4\pi t}}\frac{2}{\pi}\int_0^\infty dk\, e^{-k^2 t}\, k\, t \sum_{j=1/2}^{\infty} 2(2j+1)\,\delta_f(k), \quad (29)$$

where $\delta_f = \delta_+ + \delta_-$ (see appendix) and $j = 1/2, 3/2, ...$ in three dimensions. (The factor of four comes from the trace over spinor indicies.) Applying (23) gives

$$\Delta W_{(1/2)} = \frac{4\beta}{\pi}\int_0^\infty dk \sum_{j=1/2}^{\infty} 2(2j+1)\,\delta_f(k) - \frac{8\beta}{\pi}\int_0^\infty dk \sum_{j=1/2}^{\infty} 2(2j+1)\,\delta_f(k)\sum_{n=1}^{\infty}(-1)^n e^{-n\beta k}. \quad (30)$$

Then, summing over n we have

$$\Delta W_{(1/2)} = \frac{4\beta}{\pi}\int_0^\infty dk \sum_{j=1/2}^{\infty} 2(2j+1)\,\bar{\delta}_f(k) - \frac{8\beta}{\pi}\int_0^\infty dk \sum_{j=1/2}^{\infty} 2(2j+1)\,\frac{\delta_f(k)}{e^{\beta k}+1}. \quad (31)$$

Of course, we could have guessed this result from looking at (25), taking into account the properties of spinors. It is fairly simple to derive the above equation using densities of states in the same way as in [12], using the spinor phase shift. Thus,

$$\Delta W^N_{(1/2)} = \frac{4\beta}{\pi}\int_0^\infty dk\left(\sum_{j=1/2}^{\infty} 2(2j+1)\,\delta_f(k) - 2\sqrt{\pi}B_1 k - \pi B_{3/2} - \frac{\sqrt{\pi}B_2}{\sqrt{(k^2+M_{IR}^2)}}\right) - \frac{2\beta B_2}{\sqrt{\pi}}\, logM_{IR}^2, \quad (32)$$

$$\Delta W^T_{(1/2)} = -\frac{8\beta}{\pi}\int_0^\infty dk \sum_{j=1/2}^{\infty} 2(2j+1)\,\frac{\delta_f(k)}{e^{\beta k}+1}. \quad (33)$$

III. THICKER WALLS

The phase shift method works equally well for any bubble profile, although a differential equation must be solved numerically to find the phase shift. In the general case, one could consider the difference between the phase shifts and thin wall phase shift and then add this correction onto the effective action numerically. Alternatively, it is possible to construct an approximation scheme by perturbing about the thin wall case. For example, consider the scalar boson, for which we must solve

$$-r^{-2}(r^2u')' + m^2u + \frac{l(l+1)}{r^2}u - k^2u = -Vu, \tag{34}$$

where M is given by equation (2) and V is the correction due to a thicker wall, where we define R such that $\int_0^\infty r^2V(r)dr = 0$. The Green's function is

$$G(r,r') = -\begin{cases} kA^{-1}u_1(r)u_2(r') & r < r' \\ kA^{-1}u_2(r)u_1(r') & r > r' \end{cases} \tag{35}$$

and

$$u_1(r) = \begin{cases} j_l(k'r) & r < R \\ Aj_l(kr) - Bn_l(kr) & r > R \end{cases};\ u_2(r) = \begin{cases} Cj_l(kr) + Dn_l(kr) & r < R \\ n_l(k'r) & r > R \end{cases}, \tag{36}$$

where $k' = \sqrt{k^2 - M^2}$ and $j_l(z) = \sqrt{\frac{\pi}{2}}z^{(1-d)/2}J_{l+(d-1)/2}(z)$ in $d+1$ dimensions.

One then imposes $u \propto j_l$ as $r \to 0$ and $u \propto Aj_l - B'n_l$ as $r \to \infty$. Then the solution is

$$u = u_1 - \int_0^\infty G(r,r')V(r')u(r')\, r'^2\, dr'. \tag{37}$$

Therefore, as $r \to \infty$

$$u \to u_1 + kA^{-1}u_2 \int_0^\infty V(r')u_1^2(r')\, r'^2\, dr'. \tag{38}$$

From this it is possible to show that the correction to the phase shift is

$$\Delta\delta = -\frac{k}{A^2 + B^2}\int_0^\infty V(r)u_1^2(r)\, r^2\, dr. \tag{39}$$

Then, assuming that the Bessel functions change little as r varies over the bubble wall, they can be Taylor expanded about R, giving

$$\Delta\delta \approx -\frac{k}{A^2 + B^2}\, 2j_l(k'R)k'j_l'(k'R)\int_0^R V(r)\, r^3\, dr, \tag{40}$$

where the continuity of u_1 at the bubble wall has been used. For the scalar case,

$$B = -kR^2(k\, j_{l-1}(kR)\, j_l(k'R) - k'\, j_l(kR)\, j_{l-1}(k'R)) \tag{41}$$

and

$$A = -kR^2(k\, n_{l-1}(kR)\, j_l(k'R) - k'\, n_l(kR)\, j_{l-1}(k'R)). \tag{42}$$

The above formula can then be substituted into the thermal part of (25), summing over l and integrating over k.

IV. RESULTS

The thermal one loop effective action was calculated numerically for fermions. The phase shift is substituted into equations (32) and (33), where it is convenient to change variables $k \to z = kR$ for the step potential. Then the non-thermal part has only one parameter $\eta = m_f^2 R^2$, where m_f is the mass of the fermion. The thermal part has parameters η and βm_f (since $\beta / R = \beta m_f / \sqrt{\eta}$) as independent variables.

We work with equations (32) and (33) using a numerical package. For each value of z the function is summed over l (or j), with l increasing up to a given L until the value of the function at z converges. The thermal part of the integral converges due to the exponential damping terms. When considering the non-thermal part of the function, we must check that the integrand has the correct k dependence after making the subtraction of the divergent quantities from the sum over the phase shift. This is a good check verifying that the heat kernel coefficients are correct.

We integrate up to Z chosen to obtain the required accuracy. (All results are accurate to 1%). For large values of η (and small βm_f), larger values of Z and L are needed to give convergence. The *arctan* function (from the phase shift) has problems with branches for large values of η (whenever δ hits $\pm\pi$), requiring numerical glueing of the phase shift.

The non-thermal part of the fermion effective action can be written $\beta m_f F(\eta)$ (equation (32) with the change of variables mentioned earlier). Numerically, $F(\eta)$ fits well to a power law dependence on η, giving, in the original variables,

$$\Delta W^N_{(1/2)} = -1.51 \beta m_f^3 R^2 + 0.32 \beta m_f^4 R^3 + \frac{\beta}{24\pi} m_f^4 R^3 log \frac{m_f^2}{\mu^2}, \tag{43}$$

where we have set $M_{IR} = m_f$. The full effective action for fermions plotted against η is in figure 1 for various values of the parameter βm_f (with $\mu = m_f$).

V. DISCUSSION

We have presented a simple method the compute the prefactor in the expression for the bubble nucleation rate, applying this to infinitely thin walls. Analytic corrections, due to thicker walls are considered, by perturbing about the thin wall case. These corrections do not affect the B_1 heat kernel coefficient and one can assume that the correction to B_2 is small. Thus, only adding corrections to the thermal part of the effective action should be a good approximation.

The fermion contribution generally enhances the rate, but for large βm_f it does not and becomes suppressive. This is due to opposite signs in the thermal and non-thermal parts of the effective action (for fermions). In fact, at $\beta m_f = 5.0$ (see figure 1), the sign of the action changes for various values of $\eta = m_f R$ (for a fixed bubble wall radius). If we chose $\mu = m_t$ and $m_f = m_t$ (the mass of the top quark) as in [10], then the log term cancels for the fermion determinant. In [10] they find a negatve contribution (enhance), whereas at least for the step potential background, we find it can also be positive (suppress).

In the context of the electroweak phase transition, we would also like to consider vector bosons. This requires the use of vector spherical harmonics and the relevant boundary conditions to calculate the phase shift. Then a full treatment of all particle species at the phase transition can be worked out in the thin wall approximation, including analytic expressions for corrections due to thicker walls.

The renormalisation scale can be set by imposing conditions on the effective potential [12]. In the case of bubble nucleation, the bubble wall radius is found by extremizing the tree level action containing the free energy and surface tension. We propose choosing the radius by extremizing the one loop corrected effective action. This would be equivalent to renormalising the radius and surface tension directly, without knowledge of the tree level potential.

VI. APPENDIX

Here we present the calculation of the spinor phase shift. One must use the Dirac equation separated into radial and angular coordinates and also be careful in the way one works out the eigenvalues of the problem. Starting with

$$(i\gamma \cdot \delta - m)\psi_+ = \lambda\psi_-, \tag{44}$$

$$(i\gamma \cdot \delta + m)\psi_- = -\lambda\psi_+, \tag{45}$$

so that upon squaring the Dirac equation we get the Klein Gordon equation. The radial components are then [16]

$$(E \mp m)g_\pm + f'_\pm + (j + \frac{3}{2})\frac{f_\pm}{r} = \pm\lambda g_\mp, \tag{46}$$

$$(E \pm m)f_\pm - g'_\pm - (j + \frac{3}{2})\frac{g_\pm}{r} = \mp\lambda f_\mp, \tag{47}$$

where

$$\Psi_\pm = \begin{pmatrix} ig_\pm Y \\ f_\pm \sigma_r Y \end{pmatrix}, \tag{48}$$

for $j = l + 1/2$. The solutions are spherical Bessel functions $f_\pm = A_\pm j_{j+\frac{1}{2}}$ and $g_\pm = C_\pm j_{j-\frac{1}{2}}$ (which can be seen easily by substituting (46) into (47)) where the constants are as yet undetermined. On substituting the power series for the Bessel functions into (46) and (47) one finds the relations

$$(E \mp m)C_\pm + A_\pm k \mp \lambda C_\mp = 0, \tag{49}$$

$$(E \pm m)A_\pm + C_\pm k \pm \lambda A_\mp = 0. \tag{50}$$

The correct eigenvalue problem requires mixed boundary conditions [19] (also for a self-adjoint action) leading to $f_+ + g_+ = f_- + g_- = 0$ as $r \to \infty$. It is also possible to show that there is a symmetry among the solutions such that $f_+ \leftrightarrow f_-$ and $g_+ \leftrightarrow -g_-$. For the calculation we set $E = 0$ and impose the boundary conditions with the above symmetries, noting that $m \to 0$ as $r \to \infty$. Using this information, it is possible to show that there are two solutions

$$\tan\,\delta = \frac{B_+}{A_+}, \tag{51}$$

for $A_+ = \pm C_+$ and $B_+ = \pm D_+$, where $B_\pm$ and $C_\pm$ are the corresponding constants for $n_{j\pm\frac{1}{2}}$, the irregular solution, which are spherical Neumann functions.

Then matching the wavefunction at the junction and performing a tedious amount of algebra, we get the phase shifts,

$$\tan\,\delta_\pm = \frac{(k \pm m)\,J_j(kR)\,J_{j+1}(k'R) - k'\,J_j(k'R)\,J_{j+1}(kR)}{(k \pm m)\,N_j(kR)\,J_{j+1}(k'R) - k'\,J_j(k'R)\,N_{j+1}(kR)}. \tag{52}$$

where all symbols have their usual meanings as in the rest of the paper.

The $j = l - 1/2$ modes have symmetry such that $f_\pm \leftrightarrow g_\pm$ changes the boundary conditions and equations into the $j = l + 1/2$ case. Thus we have the above phase shifts with a total degeneracy $2(2j + 1)$, where $j = 1/2, 3/2, ...$ All results can be generalised to the four dimensional case.

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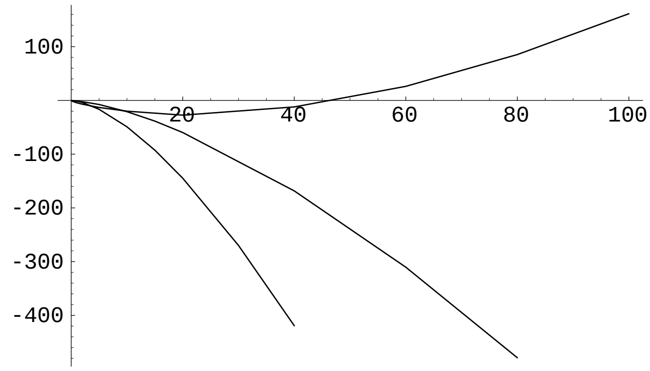

FIG. 1. The full fermionic thermal effective action $\Delta W_{(1/2)}$ with the choice $\mu = m_f$, plotted against η for various values of βm_f (from bottom to top); 0.5, 1.0 & 5.0.